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LARGE DEFORMATIONS AND STRESSES OF A THIN, HIGHLY ELASTIC, TOROIDAL SHELL UNDER INTERNAL PRESSURE

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Abstract—Behavior of highly elastic thin-walled torus under internal pressure is studied. Initial and largely deformed states are described, based on experiments. Physics (stresses, physical constants) of the problem is calculated from them. Nonlinearity is characterized by finding a range of initial shapes corresponding to a deformed state.

NOTATION

Quantities associated with the initial state are designated by lower case letters. Capital ones characterize the deformed state. A subscript indicates a material point. Superscripts ^{m} and ^{p} refer to meridians and parallels respectively. Dimensional quantities, denoted by bars, are divided by characteristic factors to be made dimensionless.

Dimensionless variables

C_1, C_2	physical constants for the material
d, D	distances of mid-points of meridional cross-sections, in horizontal plane, to axis of rotation
h	shell thickness
I_1, I_2	strain invariants
K	ratio of physical constants
l, L	modulus of position vectors, taken from mid-points
ΔL^m	radial difference
N^m, N^p	membrane stresses
o, O	mid-points of meridional cross-sections in horizontal plane
p	uniform internal pressure
r, R	distances from axis of rotation to point of torus
R^m	radius of curvature of meridional cross-section
T	point where meridional cross section has an horizontal tangent
x, X	horizontal cartesian coordinates taken from mid-points
y, Y	vertical cartesian coordinates taken from mid-points
$\lambda^m, \lambda^p, \lambda$	extension ratios
ϕ, Φ	angles of position vectors with vertical
Θ	angle of normal to deformed meridional cross section with vertical
$\Theta - \phi$	angle of rotation of membrane

INTRODUCTION

THE solution of the problem of toroidal shells of circular cross section, under internal pressure, by means of the linear membrane theory, leads to continuous stress distributions, while the displacements are discontinuous [1]. Application of the linear bending theory is appropriate to remove those deformational inconsistencies [2, 3], for a certain range of parameters determined by Reissner [4]. He also showed that, for sufficiently thin shells, use of a geometrically non linear membrane theory, is more appropriate to ensure continuity of displacements. Such a type of analysis of the problem has been given by Jordan [5] and Sanders Jr. and Liepins [6], for small strains. The stress distributions they obtained differ little from the one given by the linear membrane theory.

In the present paper, actual large continuous displacements, rotations and deformations are obtained, for a pressurized thin-walled torus made of highly elastic material, based on experiments. Following the method set forth by Green and Zerna [7], the membrane stress distributions necessary to maintain the equilibrium of the deformed shape are calculated. They differ considerably from the one obtained for small strains. Those geometrically nonlinear results are substituted in the constitutive relations, derived by Rivlin and Thomas [8], for a thin shell material having a stored-energy function of the Mooney form [9]. This physically nonlinear treatment of the problem leads to a ratio for the physical constants in good agreement with previous work [8]. The nonlinear character of the problem is finally shown by obtaining a range of initial shapes corresponding to the deformed state studied.

EXPERIMENTAL

The experimental part of the present work consists in measuring positions and shapes of a torus acted upon by uniform internal pressures. The torus, made of vulcanized, natural rubber, was of uniform thickness, about 1 mm thick. To use a conventional reference state, the torus was initially slightly inflated to present an unstrained meridional cross section approximately circular. Then, 35 material points covering the upper symmetrical half of a meridian, where marked with ink on the surface of the torus. For clarity, only one out of two points is shown on Fig. 1. Their position, with respect to the vertical axis of rotation $\bar{r}$, and the horizontal plane of symmetry $\bar{y}$, was then measured with travelling microscopes fitted with verniers which could be read to 0·01 mm. The internal pressure was then increased very slowly to a state of equilibrium at which the torus had a largely displaced and deformed shape. The position $\bar{R}$ and $\bar{Y}$ of the same 35 material points was again measured as described above. Accuracy of measurements was ensured by repeating the procedure. To present meaningful dimensionless results all linear dimensions are divided by $\bar{R}_T^m = 165{\cdot}882$ mm. The above described procedure gives the dimensionless coordinates r, y and R, Y shown, together with the other geometrical variables, on Fig. 1 drawn to scale.

From those values, the following dimensionless quantities, necessary for further analysis, are readily obtained. The mid-points o and O of the meridional cross sections are located in the horizontal plane of symmetry at

$$d = \tfrac{1}{2}(r_1 + r_{33}) = 1{\cdot}890 \quad \text{and} \quad D = \tfrac{1}{2}(R_1 + R_{33}) = 2{\cdot}404 \tag{1}$$

from the axis of rotation, for the initial and deformed state respectively. The ratio $D/d = 1{\cdot}270$, characterizes the large horizontal displacement of the torus. Dimensionless cartesian coordinates are then

$$x = r - d \quad \text{and} \quad X = R - D, \tag{2}$$

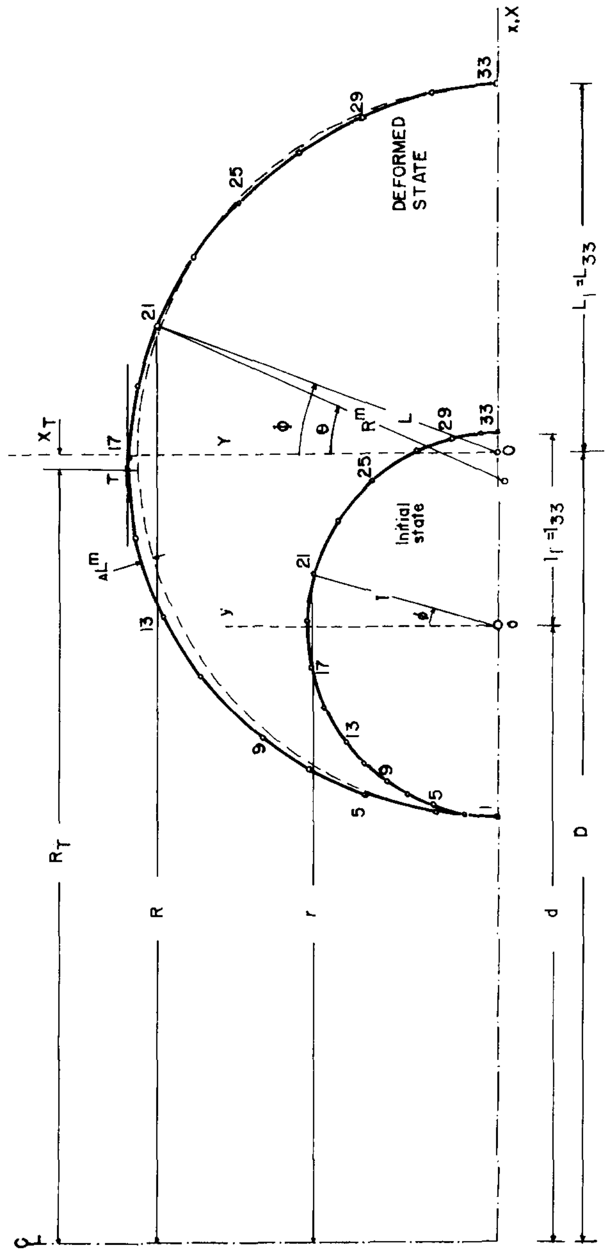

FIG. 1. Geometry and notation (drawn to scale).

taken from the mid-points o and O respectively. They permit us to obtain the angles made by the position vectors of the cross sections (originated at the mid-points) with a vertical as

$$\phi = \tan^{-1}\frac{y}{x} \quad \text{and} \quad \Phi = \tan^{-1}\frac{Y}{X}. \tag{3}$$

At a material point j,

$$\Delta\phi = \tfrac{1}{2}(\phi_{j+1} - \phi_{j-1}) \tag{4}$$

will be used. The modulus of the position vectors are

$$l = \sqrt{(x^2 + y^2)} \quad \text{and} \quad L = \sqrt{(X^2 + Y^2)}. \tag{5}$$

The ratio L/l varying between 1·86 and 1·93 characterizes the actual large radial displacements.

DEFORMATIONS

While direct use of the geometrical variables obtained in the last section, could permit an approximate description of the state of strain, it is not sufficient to calculate the stress distributions. The stresses are very sensitive to R^m, the radius of curvature of the deformed meridional cross section, mostly where the latter presents an horizontal tangent. Thus, to derive a function describing continuously and exactly R^m, the modulus of the position vector of the deformed meridional cross section, can be represented by

$$L^m(\Phi) = \sum_{i=0}^{4} A_i \sin^i \Phi. \tag{6}$$

The coefficients

$$\begin{aligned} A_0 &= 185{\cdot}51597 \\ A_1 &= -7{\cdot}99735 \\ A_2 &= -10{\cdot}12223 \\ A_3 &= 7{\cdot}99798 \\ A_4 &= 6{\cdot}03731, \end{aligned} \tag{7}$$

calculated using a least squares fit, give values within experimental precision. The variables of this problem are scaled in such a way as to be $O(1)$, thus the precision of measurement is $O(10^{-5}) \simeq |L^m - L|$. The radial difference between the deformed shape and the circle which has diameter $2L_1^m$ in common with it,

$$\Delta L^m = L^m - L_1^m \tag{8}$$

is $O(10^{-2})$, not compatible with the linear strain theory. Equation (8) is represented in Figs. 1 and 2.

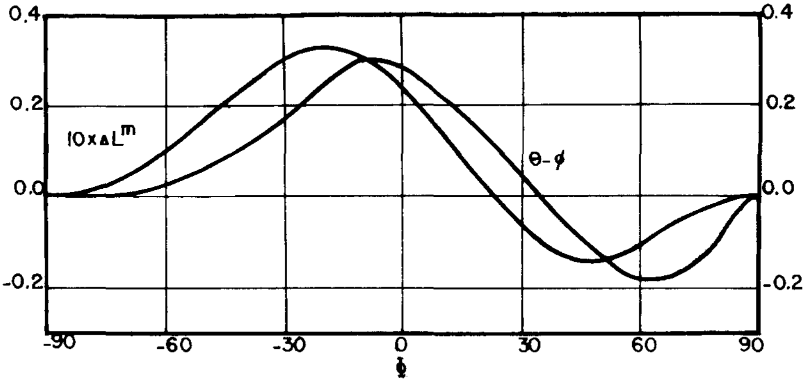

FIG. 2. Radial difference ΔL^m (exaggerated $\times$ 10) and angle of rotation of membrane $\Theta-\phi$.

The angle made by the normal to the deformed meridional cross section and a vertical is given by

$$\Theta = \tan^{-1}\frac{L\tan\Phi-\dfrac{\mathrm{d}L}{\mathrm{d}\Phi}}{L+\dfrac{\mathrm{d}L}{\mathrm{d}\Phi}\tan\Phi}. \tag{9}$$

Solving for $\Theta = 0$, $\Phi_T = -0{\cdot}03860298$ was obtained to 9 digits to ensure continuous stresses. The point T is located at

$$X_T = L_T^m \sin\Phi_T = -0{\cdot}04323 \quad \text{and} \quad R_T = D+X_T = 2{\cdot}36041,$$

with respect to O and the axis of rotation respectively. The shift of T toward the axis of rotation is $O(10^{-2})$ which is not negligible. The difference $\Theta-\phi$ is also calculated and plotted in Fig. 2 to describe the angle of rotation of the deformed meridional cross section. It is $O(10^{-1})$, not compatible with the assumption of moderately small rotations. At each material point J

$$\Delta\Theta = \tfrac{1}{2}(\Theta_{J+1}-\Theta_{J-1}), \tag{10}$$

will be needed.

The following function, describing exactly and continuously the radius of curvature of the meridional cross section,

$$R^m = \frac{\left[\left(\dfrac{\mathrm{d}L^m}{\mathrm{d}\Phi}\right)^2+L^2\right]^{\frac{3}{2}}}{2\left(\dfrac{\mathrm{d}L^m}{\mathrm{d}\Phi}\right)^2+L^2-L\left(\dfrac{\mathrm{d}^2L^m}{\mathrm{d}\Phi^2}\right)} \tag{11}$$

may then be evaluated by use of (6) and (7) and is plotted in Fig. 3. Its dimensional value at T, where $\Theta = 0$, $\bar{R}_T^m = 165{\cdot}882$ mm, is used to make dimensionless the variables of the present work. The deviation from $R^m = 1$ is of order $O(10^{-1})$. The maximum curvature does not occur in the critical range $X \simeq X_T$ but at $O(10^{-1})$ from it.

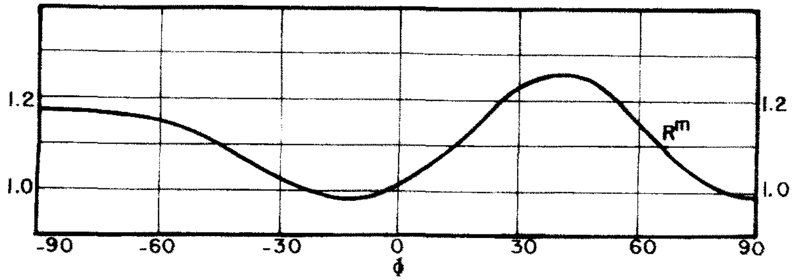

Fig. 3. Radius of curvature of meridional cross section R^m.

It is evident from the symmetry of the problem that the principal axes of strain are in the meridional, parallel and radial directions at each point of the torus. The extensions ratios for these three directions are denoted by λ^m, λ^p and λ respectively. From (4), (5), (10) and (11), the ratio of the same material fiber in the deformed and initial state is

$$\lambda^m = \frac{R^m \Delta\Theta}{l\Delta\phi} \tag{12}$$

and

$$\lambda^p = \frac{R}{r} \tag{13}$$

for a meridian and a parallel respectively.

$$\lambda = \frac{1}{\lambda^m \lambda^p}, \tag{14}$$

since for incompressible material $\lambda^m\lambda^p\lambda = 1$. λ^m and λ^p are plotted in Fig. 4, the former, varying between 3·982 and 1·290 describes an interesting state of large meridional deformations and radial displacements, the latter varying between 1·416 and 1·001, parallel deformations and horizontal displacements. The strain invariants

$$I_1 = (\lambda^m)^2 + (\lambda^p)^2 + (\lambda)^2$$

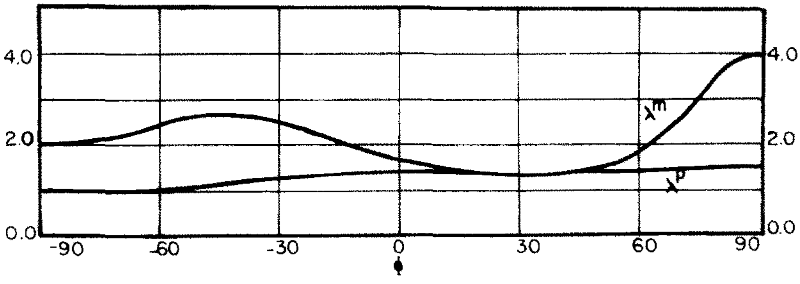

Fig. 4. Meridional λ^m and parallel λ^p extension ratios.

and

$$I_2 = (\lambda^m)^{-2} + (\lambda^p)^{-2} + (\lambda)^{-2} \tag{15}$$

are plotted in Fig. 5. Linear constitutive relations based on the hypothesis of small strain are evidently not valid for this problem. A nonlinear one will be used in the following section.

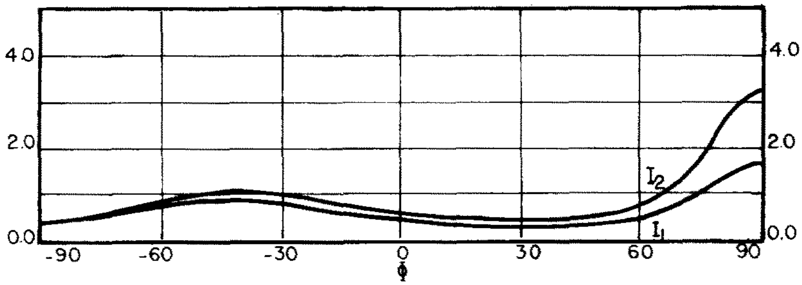

FIG. 5. Strain invariants I_1 and I_2.

STRESSES AND CONSTITUTIVE RELATIONS

In nonlinear elasticity, solutions of problems often consist in finding the system of forces which is necessary to maintain the equilibrium of an assumed position of a strained body [7]. When classical integrations are too cumbersome, this method can be used to give information on the physics of problems, by obtaining first the deformations experimentally —or motions in the case of fluids [10, 11]. As shown by Reissner [4], a nonlinear membrane theory is appropriate for a sufficiently thin-walled torus. Accordingly, the membrane stresses necessary to maintain the deformed position described above, will be obtained.

The equilibrium of the vertical force components acting on the shell segment between the point T and a point located at a distance $R = D + L^m \sin \Phi$ from the axis of rotation, gives for the dimensionless meridional membrane stress

$$N^m = \bar{N}^m / \bar{p}\bar{R}_T^m = \frac{R^2 - R_T^2}{2R \sin \Theta}, \tag{16}$$

where $\bar{p}$ is the dimensional pressure. The equilibrium of the normal force components acting on a shell element yields for the dimensionless parallel membrane stress

$$N^p = \bar{N}^p / \bar{p}\bar{R}_T^m = \frac{R}{\sin \Theta}\left(1 - \frac{N^m}{R^m}\right). \tag{17}$$

The weight of the shell is assumed to be negligible and the tangential stresses are, from the symmetry of the problem, zero. In Fig. 6 are presented the membrane stresses obtained from (16) and (17) together with the linear ones. Results obtained here differ considerably in comparisons to the ones derived for small strains. In particular, N^p become even larger than N^m in part of the shell, where it is 220 per cent greater than the linear result. In other parts N^p is only 24 per cent of the linear one.

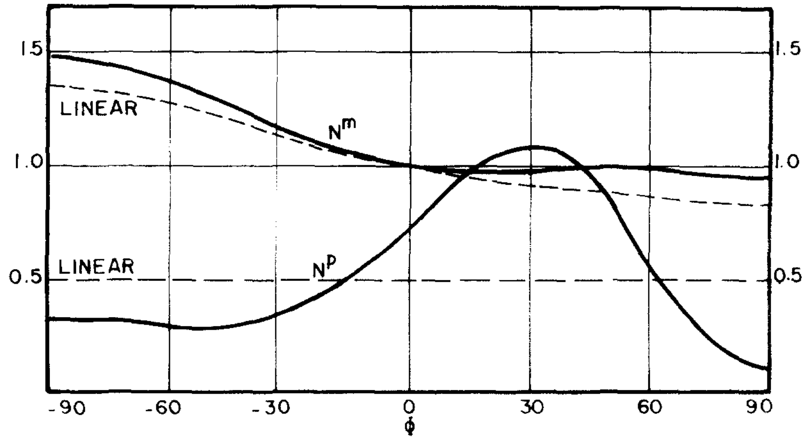

Fig. 6. Meridional N^m and parallel N^p membrane stresses.

Considering the material of the torus as highly elastic, incompressible and isotropic in its unstrained state, a stored energy function of the Mooney type

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{18}$$

may be used [9]. C_1 and C_2 are physical constants for the material. Thus from [8], the following physically nonlinear constitutive relations

$$\begin{aligned} N^m &= 2h\lambda[(\lambda^m)^2-\lambda^2][C_1+(\lambda^p)^2C_2] \\ N^p &= 2h\lambda[(\lambda^p)^2-\lambda^2][C_1+(\lambda^m)^2C_2] \end{aligned} \tag{19}$$

may be employed for the pressurized torus considered. Solving for $K = C_2/C_1$ and using (14), the ratio of the physical constants is given by

$$K = \frac{[(\lambda^p)^2(\lambda^m)^4-1]-\dfrac{N^m}{N^p}[(\lambda^p)^4(\lambda^m)^2-1]}{(\lambda^m)^2\dfrac{N^m}{N^p}[(\lambda^p)^4(\lambda^m)^2-1]-(\lambda^p)^2[(\lambda^p)^2(\lambda^m)^4-1]}. \tag{20}$$

The results obtained from (20) are shown on Fig. 7. It is found that K depends slightly on the state of deformation. For very large deformations, $\lambda^m \simeq 4$, the material tends to behave as a

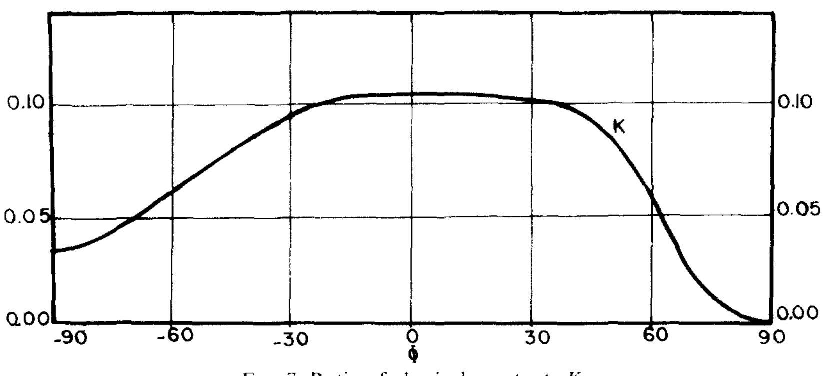

Fig. 7. Ratio of physical constants K.

so-called neo-Hookean solid, i.e. $K = 0$. In most of the section however, λ^m and λ^p vary between 1 and 2 and K is about 0·1. K is in the same range and shows a similar behavior as described in [8], implying that the constitutive equations, based on the Mooney stored-energy function are appropriate to describe physical nonlinearity of the present problem.

A largely deformed shape of a highly elastic, thin walled pressurized torus has been found in equilibrium under a system of membrane stresses. The ratio of the physical constants of the nonlinear constitutive relations, connecting the deformed to the initial state (obtained experimentally), was found in accordance with previous work. The nonlinearity of the problem can be shown by finding—through an iterative procedure—a range of initial shapes which could be connected to the deformed state, described above, by the nonlinear constitutive relations used (using same K). Figure 8 shows the location of possible mid-points o vs. the horizontal initial radius l_1. The very small deviations of l from l_1 are taken proportionally to the one obtained in the experimental case.

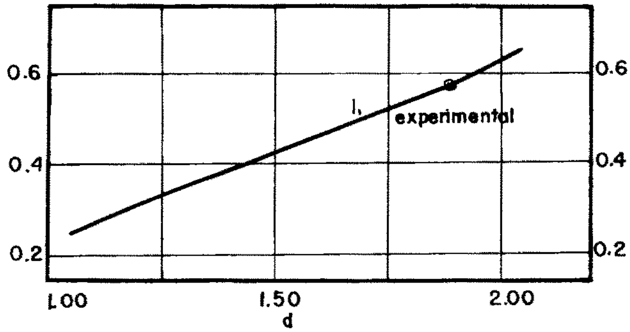

FIG. 8. Modulus of horizontal position vector l_1 vs. location of mid-point d for possible initial shapes.

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Абстракт—Исследуется поведение идеально упругого тонкостенного тора под влиянием внутреннего давления. Описывается начальное состояние и состояние при больших деформациях, основанное на экспериментах. Далее, определяется физическая сторона задачи 'напряжения, физические постоянные'. Изображается нелинейность тора путем определения круга начальных форм, соответствующих деформированному состоянию.